14.1 - Multivariable Functions

- <u>Def</u>: A <u>multivariable function</u> (of real inputs and real output) is a function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$

function's name — does something — domain — codomain (where outputs live)

  * $\text{dom}(f) = D$ in this notation
  * $\text{ran}(f) = \{f(\vec{x}) : x \in \text{dom}(f)\}$

  * If no domain is specified, we assume the biggest possible domain (i.e. the "natural domain").

- Ex) $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

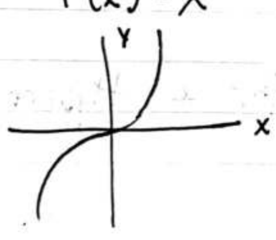
  In this case, $\text{dom}(f) = \{(x,y)^{\in \mathbb{R}^2} : \frac{x^2-y^2}{x^2+y^2} \text{ is defined}\}$.
  $= \{(x,y)^{\in \mathbb{R}^2} : x^2 + y^2 \neq 0\}$
  $= \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$

- Ex) $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$
  has the same domain as before

- Ex) $f(x,y) = \dfrac{x+y+1}{x^2-y^2}$
  $\text{dom}(f) = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 0\}$
  $= \{(x,y) \in \mathbb{R}^2 : |x| \neq |y|\}$

- <u>Def</u>: The graph of a function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ is
  $\text{graph}(f) = \{(\vec{x}, f(\vec{x})) : \vec{x} \in \text{dom}(f)\}$.

  * Ex) from Calc I: $f(x) = x^3$



* If $n=2$ (function has 2 variables) this becomes:

$$\text{graph}(f) = \{(x, y, f(x,y)) : (x,y) \in \text{dom}(f)\}$$

i.e. this is "a picture" of:

$z = f(x,y)$'s solution sets.

- Ex) What does graph(f) look like for:

$$f(x,y) = \sqrt{x^2+y^2+1} \ ?$$

$z = f(x,y)$

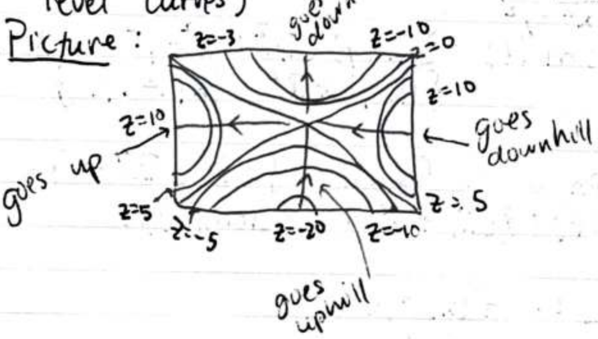
$z = \sqrt{x^2+y^2+1}$

$z^2 = x^2+y^2+1$ (for $z \geq 0$)

$-x^2-y^2+z^2=1$ ← Two-Sheet Hyperboloid

* so graph is one of the 2 sheets of the Two-Sheet Hyperboloid.

- How do we represent graph(f) for a 2-variable function?
  * Draw a contour map (or elevation map or level curves)
  * Picture:



* has cross sections for a Hyperbolic Paraboloid

- Ex) in 4-dimensions: The unit hypersphere

$$S^3 = \{\vec{x} \in \mathbb{R}^4 : \underline{|\vec{x}| = 1}\} \to |w| \le 1$$

(x, y, z, w)

* Once $w = k$ is fixed

$$\sqrt{x^2 + y^2 + z^2 + k^2} = 1$$

$$x^2 + y^2 + z^2 = 1 - k^2$$

sphere of radius $\sqrt{1-k^2}$ about the origin

* We get a movie describing the hypersphere (w = time).

$w = -1$:

$w = 0$:

$w = -\frac{1}{2}$:

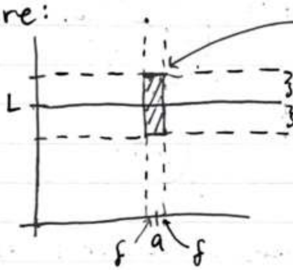
$w = \frac{1}{2}$

* Sphere gets bigger, then smaller & disappears.

14.2 - Limits and Continuity of Multivariable Functions

- In Calc I, the formal def. for a limit was:
  Def: Let f be a function and $a \in \mathbb{R}$ be a limit point of the domain of f. The limit of f as x tends to a is $L \in \mathbb{R}$ when: for $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in \mathrm{dom}(f)$ we have $|x-a| < \delta$ implies $|f(x) - L| < \varepsilon$.

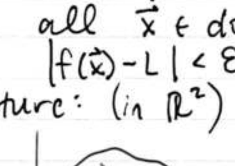
  Picture:



all of $f(x)$ lives in this rectangle (near a).

- In Calc III, the formal def. for the limit is:
  Def: Let f be a multivariable function and let $\vec{a} \in \mathbb{R}^n$ be a limit point of the domain of f. The limit of f as $\vec{x}$ tends to $\vec{a}$ is $L \in \mathbb{R}$ when: for all ~~[illegible]~~ $\varepsilon > 0$ there is a $\delta > 0$ such that for all $\vec{x} \in \mathrm{dom}(f)$ we have $|\vec{x} - \vec{a}| < \delta$ implies $|f(\vec{x}) - L| < \varepsilon$.

  Picture: (in $\mathbb{R}^2$)



can approach the point in many different ways.

* This def. is hard to use... We'll use prop. in its place.
(multivariable version of one-sided limits)

- Prop (Curves Criterion for Limits):
  - * Suppose f is a multivariable function and a is a limit point of its domain. The $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ iff for all space curves $\vec{r}(t)$ in $\mathrm{dom}(f)$ such that $\lim_{t\to\infty} \vec{r}(t) = \vec{a}$ we have $\lim_{t\to\infty} f(r(t)) = L$.
  - * Notation: $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$
    - alt.: $f(\vec{x}) \to L$ as $\vec{x} \longrightarrow \vec{a}$.

- Ex) Show that the $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ does not exist.

Consider the collection $l_{a,b}(t) = \langle at, bt \rangle$ where $(a,b) \neq (0,0)$ of lines. Observe $\lim_{t\to 0} l_{a,b}(t) = \langle 0,0 \rangle$.
For $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$, we have $f(l_{a,b}(t)) = f(at, bt)$

$$= \frac{(at)^2-(bt)^2}{(at)^2+(bt)^2} = \frac{(a^2-b^2)t^2}{(a^2+b^2)t^2} = \frac{a^2-b^2}{a^2+b^2}$$

If the limit exists, we have: $\lim_{t\to 0} f(l_{a,b}(t)) = L$ for all a, b.

$$\rightarrow \lim_{t\to 0} f(l_{a,b}(t)) = \lim_{t\to 0} \frac{a^2-b^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$$

But if $a=1, b=0$, we have $L=1$
and if $a=1=b$, we have $L=0$

The limit does not exist by the curves criterion.